

Superintegrable systems with position dependent mass

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Abstract

First order integrals of motion for Schrödinger equations with position dependent masses are classified. Eighteen classes of such equations with non-equivalent symmetries are specified. They include integrable, superintegrable and maximally superintegrable systems. Among them is a system invariant with respect to the Lie algebra of Lorentz group and a system whose integrals of motion form algebra $so(4)$. Three of the obtained systems are solved exactly.

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I. INTRODUCTION

Schrödinger equations are perfect subjects for demonstration of the richness and profundity of the concept of symmetry. In addition to symmetries with respect to continuous groups, they admit supersymmetries and hidden symmetries like the Fock symmetry of the Hydrogen atom. Moreover, just Schrödinger equations belong to the nice field of the inverse problem approach with its infinite number of symmetries and constants of motion.

The history of searching for symmetries of Schrödinger equation is both long and inspiring. The invariance of this equation w.r.t. Galilei group was in fact predicted by Sophus Lie. More exactly, Lie found the maximal continuous invariance group of the heat equation, which in the main coincides with the symmetry group of the free Schrödinger equation.

A systematic search for Lie symmetries of Schrödinger equation was carried out in papers [1], [2], [3], [4] where the maximal invariance groups of this equation with arbitrary potential were presented. The next level symmetries, i.e., the second order symmetry operators for $2d$ and $3d$ Schrödinger equation have been found in [5], [6] and [7], [8]. Symmetry operators of arbitrary order for the free Schrödinger equation had been enumerated in [9], the completed group classification of the nonlinear Schrödinger equations was presented in [10].

The extended (in particular, second order) symmetries are requested for description of systems admitting solutions in separated variables [11], integrable and superintegrable systems [12]. A relatively new field is presented by superintegrable systems with spin whose systematic investigation was started with paper [13] and continued in [14], [15], [16], [17] and [18], see also survey [12]. The subject of the research carried out in [14], [15] are systems with spin-orbit interaction while the systems with Pauli type interactions are studied in [16], [17] and [18].

Let us note that the first example of a superintegrable system with spin $1/2$ was presented earlier in paper [19]. Superintegrable systems with arbitrary spin were discussed in [20], [21], [22] and [23], the relativistic systems were elaborated in [16] and [24].

In the present paper we discuss superintegrability aspects of position dependent mass (PDM) Schrödinger equations which attract interest of many researchers and are applied in several physical systems. They are requested for description of various condensed-matter systems such as semiconductors [25], [26], quantum liquids [27] and metal clusters [28], quantum wells, wires and dots [29], [30], super-lattice band structures [31] and many, many others.

There are numerous papers devoted to exact solutions of PDM quantum mechanical prob-

lems, see, e.g., [32], [33] [34] and the references therein. The very possibility to solve exactly a mechanical or quantum mechanical equation is caused by its symmetry. And it is the reason why symmetries (and supersymmetries) of particular PDM problems were in the focus of attention of many researches, see, e.g., [34], [35], [36], [37]. However, in contrast with the case of constant mass, there are no general results concerning the group classification of generic PDM problems and complete sets of their integrals of motion. An exception is paper [38] where the generic form of PDM Schrödinger equation compatible with the Galilei invariance postulate is presented. In addition, starting with classical systems defined in curved spaces, the extended and well defined class of PDM superintegrable models was found and studied in papers [39], [40], [41], [42] and [43], see also the references cited therein.

In the present paper we start a systematic investigation of integrable and superintegrable systems with position dependent mass. As the first step we classify PDM Schrödinger equations which admit the first order integrals of motion. Since these equations are defined up to two arbitrary functions, the level of complexity of the classification problem is compatible with the one characterizing papers [13] and [17] where the first order integrals of motion for the Schrödinger equations with constant masses and matrix potentials were classified.

It will be shown that there exist 18 non-equivalent classes of PDM Schrödinger equations with non-trivial integrals of motion. Among them there are superintegrable systems invariant with respect to the Lie algebras of Lorentz group $SO(1,3)$ and rotation group $SO(4)$. We present solutions of these systems which appears to be exactly solvable. Moreover, the spectra of the related Hamiltonians can be found algebraically like in the case of the Hydrogen atom.

II. DETERMINING EQUATIONS

Let us consider a stationary PDM Schrödinger equation

$$\hat{H}\psi = E\psi, \quad (1)$$

where H is the following generic Hamiltonian

$$\hat{H} = p_a f(\mathbf{x}) p_a - V(\mathbf{x}) = -\partial_a f(\mathbf{x}) \partial_a - V(\mathbf{x}). \quad (2)$$

Here $\mathbf{x} = (x^1, x^2, x^3)$, $p_a = -i\partial_a$, $V(\mathbf{x})$ and $f(\mathbf{x}) = \frac{1}{2m(\mathbf{x})}$ are arbitrary functions associated with the effective potential and inverse effective PDM, and summation from 1 to 3 is imposed over the repeating index a .

Let us search for first order integrals of motion for equation (1), i.e., for commuting with H differential operators of first order which we write in the form:

$$Q = \frac{1}{2}(\xi^a p_a + p_a \xi^a) + \tilde{\eta} = -i(\xi_a \partial_a + \eta) \quad (3)$$

where $\eta = \frac{1}{2}\partial_a \xi^a + i\tilde{\eta}$, ξ^a and η are functions of \mathbf{x} .

By definition, operators H and Q should commute each other:

$$[\hat{H}, Q] \equiv \hat{H}Q - Q\hat{H} = 0. \quad (4)$$

Since the commutator of first order differential operators is again a first order differential operator, the complete set of integrals of motion (3) form a basis of a Lie algebra. Thus in fact we are searching for invariance algebras of equation (1). Moreover, these algebras can be integrated to local Lie groups.

Relation (4) has to be treated as an operator equation. It means that this differential expression should nullify any twice differentiable function. In other words, to solve this equation it is necessary to equate to zero the coefficients for all distinct differentials in (4). In this way, after calculating the commutator, we obtain the following system of determining equations:

$$\xi^c f_c \delta_{ab} - f(\xi_a^b + \xi_b^a) = 0, \quad (5)$$

$$-\xi^i f_{ai} + f_i \xi_i^a + f \xi_{cc}^a + 2f \eta_a = 0, \quad (6)$$

$$f_a \eta_a + f \eta_{aa} - \xi^a V_a = 0. \quad (7)$$

Here δ_{ab} is the Kronecker symbol, and subindices denote derivations with respect to the corresponding spatial variable: $\eta_a = \partial_a \eta$, etc.

The system of equations (5)–(7) gives the necessary and sufficient conditions for commutativity of operators H and Q . It is overdetermined and includes ten equations for six unknowns.

Let us evaluate the determining equations. First we note that equation (5) can be decoupled to two subsystems:

$$\xi_a^b + \xi_b^a = \frac{2}{3} \delta_{ab} \xi_i^i, \quad (8)$$

$$3\xi^i f_i = 2f \xi_i^i \quad (9)$$

which are nothing but the traceless part and trace of (5).

Equation (8) defines the 3d conformal Killing vector, whose general form is given by the following expression (see, e.g., [44])

$$\xi^a = \lambda^a x^n x^n - 2x^a \lambda^n x^n + \mu^c \varepsilon^{abc} x^b + \omega x^a + \nu^a \quad (10)$$

where the Greek letters denote arbitrary real constants.

Thus we already know the general form of functions ξ^a which are second order polynomials in x^a . To find the remaining function η needed to fix Q it is sufficient to differentiate (9) w.r.t. x_a and compare the result with (6). In this way, using equations (8) and the following identities

$$\xi_{aa}^i = 2\lambda_i, \quad \xi_{ia}^i = -6\lambda_a$$

we immediately find, that

$$\eta = -3\lambda_a x_a + c \quad (11)$$

where c is a constant.

If relations (10), (11) and (9) hold then (6) and (8) are satisfied identically, while the remaining equations (7) and (9) are reduced to the following form:

$$\xi^i f_i = 2(\omega - 2\lambda_a x_a) f, \quad (12)$$

$$\xi^i V_i = -3\lambda_i f_i \quad (13)$$

where ξ^i are functions given in (10). Thus our classification problem is reduced to finding the general solution of equations (12) and (13) for unknowns f and V , where ξ^i are polynomials (10).

III. NONEQUIVALENT VERSIONS OF EQUATIONS (12) AND (13)

Equations (12) and (13) include ten arbitrary parameters. Our next task is to specify values of these parameters which correspond to non-equivalent versions of these equations.

To define equivalence relations for solutions of system (12), (13) we note that in accordance with (3), (10), (11) the generic first order integral of motion for equation (1) can be represented as the following linear combination:

$$Q = \lambda^i K^i + \mu^i J^i + \omega D + \nu^i P^i + c \quad (14)$$

where

$$\begin{aligned} P^i &= p^i = -i \frac{\partial}{\partial x_i}, \quad J^i = \varepsilon^{ijk} x^j p^k, \\ D &= x^n p^n - \frac{3i}{2}, \quad K^i = x^n x^n p^i - 2x^i D, \end{aligned} \quad (15)$$

and c is a constant which is not essential and will be chosen as zero.

Operators (15) satisfy the following commutation relations:

$$\begin{aligned}
[P^a, P^b] &= 0, \quad [P^a, J^b] = i\varepsilon_{abc}P^c, \\
[J^a, J^b] &= i\varepsilon_{abc}J^c, \quad [D, J_a] = 0, \\
[D, P^a] &= iP^a, \quad [D, K^a] = -iK^a, \\
[K^a, J^b] &= i\varepsilon_{abc}K^c, \quad [K^a, K^b] = 0, \\
[K^a, P^b] &= 2i(\delta^{ab}D - \varepsilon_{abc}J^c)
\end{aligned} \tag{16}$$

where ε_{abc} is the Levi-Civita symbol. In other words, they form a basis of the Lie algebra of conformal group $C(3)$ defined in the $3d$ Euclidean space.

We see that the required integrals of motion should belong to algebra $c(3)$ whose basis elements are given by equation (15). However, if all parameters λ^a, ν^a, μ^a and ω are arbitrary, then the determining equations (12) and (13) are compatible iff $f(\mathbf{x}) = 0, V(\mathbf{x}) = Const$, and equation (1) is trivial. To obtain a non-trivial equation (1) it is necessary to impose some constraints on these parameters and reduce the algebra $c(3)$ to one of its subalgebras (we remind that the first order integrals of motion have to form a Lie algebra [11]).

Thus to fix all non-equivalent versions of equations (12) and (13) it is necessary to find all non-equivalent sets of parameters λ^a, ν^a, μ^a and ω , which correspond to non-equivalent subalgebras of algebra $c(3)$. The optimal system of such subalgebras can be defined up to the group of internal isomorphisms, which is group $C(3)$.

Happily, the optimal system of subalgebras of $c(3)$ are well known. To enumerate them we note that algebra $c(3)$ is isomorphic to $so(1,4)$, i.e., to the Lie algebra of the Poincaré group in $1+4$ dimensional space. This isomorphism can be fixed by the following formulae

$$M^{ab} = \varepsilon^{abc}J^c, \quad M^{0a} = \frac{1}{2}(K^a + P_a), \quad M^{4a} = \frac{1}{2}(K^a - P_a), \quad M^{04} = D \tag{17}$$

where $M^{\mu\nu}$ with $\mu, \nu = 0, 1, 2, 3, 4$ are basis elements of algebra $so(1,4)$, satisfying the following relations:

$$[M^{\mu\nu}, M^{\lambda\sigma}] = i(g^{\mu\sigma}M^{\nu\lambda} + g^{\nu\lambda}M^{\mu\sigma} - g^{\mu\lambda}M^{\nu\sigma} - g^{\nu\sigma}M^{\mu\lambda}) \tag{18}$$

where $g^{\mu\nu} = diag(1, -1, -1, -1)$. One can make sure that commutation relations (18) are consequences of (16), (17), and vice versa, relations (16) follows from (18) and (17).

Optimal subalgebras of algebra $so(1,4)$ have been classified in paper [45]. Using this classification and applying isomorphism (17) we obtain the following list of non-equivalent subalgebras:

One dimension subalgebras:

$$\begin{aligned} & \langle M^{21} \rangle, \quad \langle M^{43} + \alpha M^{21} \rangle, \quad 0 \leq \alpha \leq 1, \quad \langle M^{43} - M^{03} \rangle, \\ & \langle M^{21} \cos c + M^{03} \sin c \rangle, \quad 0 < c \leq \frac{\pi}{2}, \quad \langle M^{43} - M^{03} + M^{21} \rangle; \end{aligned} \quad (19)$$

Two dimension subalgebras:

$$\begin{aligned} & \langle M^{43}, M^{21} \rangle, \quad \langle M^{42} - M^{02}, M^{41} - M^{01} \rangle, \quad \langle M^{04}, M^{21} \rangle, \\ & \langle M^{21}, M^{43} - M^{03} \rangle, \quad \langle M^{43} - M^{03}, M^{40} + \alpha M^{21} \rangle; \end{aligned} \quad (20)$$

Three dimension subalgebras:

$$\begin{aligned} & \langle M^{43} - M^{12}, M^{42} - M^{31}, M^{41} - M^{23} \rangle, \\ & \langle M^{12} \cos c - M^{04} \sin c, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \quad \langle M^{12}, M^{23}, M^{31} \rangle, \\ & \langle M^{21}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \quad \langle M^{04}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \\ & \langle M^{01}, M^{02}, M^{12} \rangle, \quad \langle M^{43} + M^{03}, M^{42} + M^{02}, M^{41} + M^{01} \rangle, \\ & \langle M^{43} + M^{03} + M^{21}, M^{42} + M^{02}, M^{41} + M^{01} \rangle, \quad \langle M^{04}, M^{12}, M^{43} - M^{03} \rangle; \end{aligned} \quad (21)$$

four dimension subalgebras:

$$\begin{aligned} & \langle M^{43}, M^{21}, M^{42} - M^{31}, M^{41} + M^{32} \rangle, \quad \langle M^{04}, M^{12}, M^{31}, M^{23} \rangle, \\ & \langle M^{12}, M^{04}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \quad \langle M^{12}, M^{43} - M^{03}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \\ & \langle M^{04}, M^{43} - M^{03}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \quad \langle M^{43}, M^{01}, M^{02}, M^{12} \rangle; \end{aligned} \quad (22)$$

five dimension subalgebra:

$$\langle M^{04}, M^{12}, M^{43} + M^{03}, M^{42} + M^{02}, M^{41} + M^{01} \rangle; \quad (23)$$

six dimension subalgebras:

$$\begin{aligned} & \langle M^{12}, M^{31}, M^{23}, M^{43} - M^{03}, M^{42} - M^{02}, M^{41} - M^{01} \rangle, \\ & \langle M^{31}, M^{23}, M^{12}, M^{01}, M^{02}, M^{03} \rangle, \langle M^{41}, M^{42}, M^{43}, M^{12}, M^{31}, M^{23} \rangle; \end{aligned} \quad (24)$$

seven dimension subalgebra:

$$\langle M^{41}, M^{12}, M^{31}, M^{43} + M^{03}, M^{42} + M^{02}, M^{41} + M^{01} \rangle; \quad (25)$$

and ten dimension subalgebra:

$$\langle M^{41}, M^{42}, M^{43}, M^{01}, M^{02}, M^{03}, M^{12}, M^{31}, M^{23}, M^{04} \rangle \quad (26)$$

where $M^{\mu\nu}$ with $\nu = 0, 1, \dots, 4$ are operators defined by equations (17).

Formulae given above repeat (but also slightly correct) the classification results presented in paper [45]. Namely:

- The one dimension algebra spanned on $M^{43} - \alpha M^{01} = A + \alpha K$ presented in [45], is equivalent to another one dimensional algebra, i.e., $\langle M^{21} \cos c - M^{03} \sin c \rangle$ presented there. We change it by the correct representative $\langle M^{43} + \alpha M^{21} \rangle$, see (19).
- In contrast with [45] we do not fix particular values $\alpha = 0, 1$ and $c = \frac{\pi}{2}$ in (19) since they do not correspond to special integrals of motion.
- To simplify the form of functions presented in the following Table 2 we change some algebras presented in [45] by another but equivalent ones. In particular, the one dimensional algebra $\langle M^{32} - M^{03} \rangle$ is replaced by the equivalent algebra $\langle M^{43} - M^{03} \rangle$.

Any of the enumerated subalgebras corresponds to a system of the determining equations (12) and (13). More exactly, any of the one dimensional algebras generates a system of such equations, the two dimension algebras generate pairs of such systems, and so on. And our classification problem is decoupled to 34 subproblems corresponding to subalgebras (19)–(26). The related functions ξ^i and parameters λ^a, ω are easily recovered using definitions (10), (14) and (17). They are presented in Table 1.

Table 1. Functions ξ^a and η corresponding to basis elements of algebra $\mathfrak{so}(1,4)$.

No	Operators	ξ^1	ξ^2	ξ^3	η	Non-zero parameters in (14)
1	iM_{43}	$x_1 x_3$	$x_2 x_3$	$\frac{s_3+1}{2}$	$\frac{3x_3}{2}$	$\lambda_3 = -\nu_3 = \frac{1}{2}$
2	iM_{42}	$x_1 x_2$	$\frac{s_2+1}{2}$	$x_2 x_3$	$\frac{3x_2}{2}$	$\lambda_2 = -\nu_2 = \frac{1}{2}$
3	iM_{41}	$\frac{s_1+1}{2}$	$x_1 x_2$	$x_1 x_3$	$\frac{3x_1}{2}$	$\lambda_1 = -\nu_1 = \frac{1}{2}$
4	iM_{40}	x_1	x_2	x_3	0	$\omega = 1$
5	iM_{32}	0	x_3	$-x_2$	0	$\mu_1 = -1$
6	iM_{31}	x_3	0	$-x_1$	0	$\mu_2 = 1$
7	iM_{21}	x_2	$-x_1$	0	0	$\mu_3 = -1$
8	iM_{03}	$x_1 x_3$	$x_2 x_3$	$\frac{s_3-1}{2}$	$\frac{3x_3}{2}$	$\lambda_3 = \nu_3 = \frac{1}{2}$
9	iM_{02}	$x_1 x_2$	$\frac{s_2-1}{2}$	$x_2 x_3$	$\frac{3x_2}{2}$	$\lambda_2 = \nu_2 = \frac{1}{2}$
10	iM_{01}	$\frac{s_1-1}{2}$	$x_1 x_2$	$x_1 x_3$	$\frac{3x_1}{2}$	$\lambda_1 = \nu_1 = \frac{1}{2}$

where $s_a = 2x_a^2 - r^2$, $a = 1, 2, 3$. Functions ξ^i corresponding to basis elements of subalgebras enumerated above are evident linear combinations of ones given in the table.

IV. SOLUTION OF DETERMINING EQUATIONS

Thus to find all non-equivalent first order integrals of motion for equation (1) it is necessary and sufficient to solve determining equations (12) and (13) where ξ^a are polynomials specified in (10). Moreover, it is sufficient to go over the reduced versions of ξ^a which correspond to the subalgebras of algebra $c(3)$ enumerated in (19)–(26). The explicit forms of these functions corresponding to basis elements (17), (15) are presented in Table 1.

Let us start with the one dimension subalgebra spanned on M^{03} (we set $c = \frac{\pi}{2}$ in (19)). The corresponding functions ξ^a , η and nonzero parameter λ_3 are presented in the eighth line of Table 1. Substituting these functions into equation (12) we obtain the following equation for f :

$$2x_3(x_1f_1 + x_2f_2 + x_3f_3) - (r^2 + 1)f_3 = 4x_3f \quad (27)$$

whose general solution has the following form:

$$f(\mathbf{x}) = \tilde{r}^2 F\left(\frac{x_2}{x_1}, \frac{r^2 - 1}{\tilde{r}}\right). \quad (28)$$

Here $\tilde{r}^2 = x_1^2 + x_2^2$, and $F(., .)$ is an arbitrary function of its arguments.

The next step is finding the corresponding potential $V(\mathbf{x})$ which solves the related equation (13), i.e.,

$$2x_3(x_1V_1 + x_2V_2 + x_3V_3) - (r^2 + 1)V_3 = 3f_3 \quad (29)$$

where f is the function presented in (28). This linear inhomogeneous equation is solved by the following function:

$$V(\mathbf{x}) = 3\tilde{r}D_2F + \tilde{F}\left(\frac{x_2}{x_1}, \frac{r^2 - 1}{\tilde{r}}\right) \quad (30)$$

where D_2F is the derivative of function F with respect to its second argument $\frac{r^2-1}{\tilde{r}}$ and $\tilde{F}(., .)$ is one more arbitrary function of $\frac{x_2}{x_1}$ and $\frac{r^2-1}{\tilde{r}}$.

Thus Hamiltonian (2) admits integral of motion M_{03} iff function f and potential V are given by equations (28) and (30). We see that there exist a rather extended class of Hamiltonians (2)

admitting M_{03} . The corresponding free elements f and V depend on two arbitrary functions and are presented by equations (28) and (30).

Let us specify these functions by requiring that the corresponding Hamiltonians admit the additional integral of motion M_{21} . In this case functions (28) and (30) have to satisfy the following additional equations:

$$x_2 f_1 - x_1 f_2 = 0, \quad x_2 V_1 - x_1 V_2 = 0,$$

see (12), (13) and line 7 of Table 1. In other words, functions (28) and (30) have to be independent on their first argument $\frac{x_2}{x_1}$. Thus functions f and V are reduced to the following forms:

$$f(\mathbf{x}) = \tilde{r}^2 F\left(\frac{r^2 - 1}{\tilde{r}}\right), \quad V(\mathbf{x}) = 3\tilde{r}F' + \tilde{F}\left(\frac{r^2 - 1}{\tilde{r}}\right). \quad (31)$$

The symbol F' in (31) denotes the derivation of function $F\left(\frac{r^2 - 1}{\tilde{r}}\right)$ with respect to its argument $\frac{r^2 - 1}{\tilde{r}}$.

If Hamiltonian (2) admits two other integrals of motion, namely, $\langle M^{03}, M^{32} - M^{02} \rangle$, then functions $f(\mathbf{x})$ (28) and $V(\mathbf{x})$ (30) should satisfy the following additional conditions:

$$\begin{aligned} 2x_2(x_1 f_1 + x_2 f_2 + (x_3 - 1)f_3) - (r^2 - 1 - 2x_3)f_2 &= 4x_2 f, \\ 2x_2(x_1 V_1 + x_2 V_2 + (x_3 - 1)V_3) - (r^2 - 1 - 2x_3)V_2 &= 3f_2. \end{aligned} \quad (32)$$

Functions (28) and (30) solve equations (32) iff they are reduced to the following special form:

$$f(x) = x_1^2 F\left(\frac{r^2 - 1}{x_1}\right), \quad V = 3x_1 F' + \tilde{F}\left(\frac{r^2 - 1}{x_1}\right). \quad (33)$$

Thus we have gone over all optimal subalgebras of dimension one and two, which include basis element M^{03} . There is only one three dimension subalgebra including M^{03} in the list (21), namely, $\langle M^{03}, M^{32} - M^{02}, M^{31} - M^{01} \rangle$. To specify Hamiltonians (2) which admit this algebra we look for functions (33) satisfying the additional constraint

$$\begin{aligned} 2x_1(x_1 f_1 + x_2 f_2 + (x_3 - 1)f_3) - (r^2 - 1 - 2x_3)f_1 &= 4x_1 f, \\ 2x_1(x_1 V_1 + x_2 V_2 + (x_3 - 1)V_3) - (r^2 - 1 - 2x_3)V_1 &= 3f_1 \end{aligned} \quad (34)$$

which are equations (12) and (13) corresponding to the difference of data from lines 6 and 10 of Table 1.

Substituting (33) into (34) and integrating the resultant equations we obtain:

$$f(x) = \mu(r^2 - 1)^2, \quad V = 6\mu r^2 + \nu \quad (35)$$

where μ and ν are arbitrary constants. These solutions are compatible with all other extensions of algebras $\langle M^{03}, M^{21} \rangle$ and $\langle M^{03}, M^{32} - M^{02} \rangle$ enumerated in (22)–(26).

In analogous way we calculate integrals of motion forming the other algebras enumerated in (19)–(26). The final classification results are presented in Table 2.

Table 2. Functions f and V in Hamiltonians (2) and the corresponding integrals of motion.

No	f	V	Integrals of motion
1	$F(\tilde{r}^2, x_3)$	$\tilde{F}(\tilde{r}^2, x_3)$	M^{21}
2	$F(x_1, x_2)$	$\tilde{F}(x_1, x_3)$	$M^{43} - M^{03}$
3	$\tilde{r}^2 F(\frac{r^2+1}{\tilde{r}}, \omega)$	$3\tilde{r}^2(q_1 D_1 - q_2 D_2)F + \tilde{F}(\frac{r^2+1}{\tilde{r}}, \omega)$	$M^{43} + \alpha M^{21}$
4	$r^2 F(\frac{\tilde{r}}{r}, r e^{\varphi \tan(c)})$	$\tilde{F}(\frac{\tilde{r}}{r}, r e^{\varphi \tan(c)})$	$M^{21} \cos c + M^{04} \sin c$
5	$F(\tilde{r}^2, x_3 - \varphi)$	$\tilde{F}(\tilde{r}^2, x_3 - \varphi)$	$M^{43} - M^{03} + M^{21}$
6	$\tilde{r}^2 F(\frac{r^2+1}{\tilde{r}})$	$3\tilde{r}F' + \tilde{F}(\frac{r^2+1}{\tilde{r}})$	M^{43}, M^{21}
7	$F(\tilde{r})$	$\tilde{F}(\tilde{r})$	$M^{21}, M^{43} - M^{03}$
8	$\tilde{r}^2 F(\ln \tilde{r}^\alpha + \varphi)$	$\tilde{F}(\ln \tilde{r}^\alpha + \varphi)$	$M^{43} - M^{03}, M^{40} + \alpha M^{21}$
9	$x_3^2 F(\frac{r^2-1}{x_3})$	$3x_3 F' + \tilde{F}(\frac{r^2-1}{x_3})$	M^{21}, M^{02}, M^{01}
10	$F(x_3)$	$\tilde{F}(x_3)$	$M^{41} - M^{01}, M^{42} - M^{02}, M^{12}$
11	$F(r^2)$	$\tilde{F}(r^2)$	M^{32}, M^{31}, M^{21}
12	$\mu \tilde{r}^2$	ν	$M^{40}, M^{21}, M^{43} - M^{03}$
13	μx_3^2	ν	$M^{12}, M^{04}, M^{42} - M^{02}, M^{41} - M^{01}$
14	μr^2	ν	$M^{04}, M^{12}, M^{31}, M^{23}$
15	$\mu((r^2 - 1)^2 + 4x_3^2)$	$6\mu r^2 + \nu$	$M^{43}, M^{01}, M^{02}, M^{21}$
16	$\mu(r^2 + 1)^2$	$6\mu r^2 + \nu$	$M^{41}, M^{42}, M^{43}, M^{21}, M^{31}, M^{32}$
17	$\mu(r^2 - 1)^2$	$6\mu r^2 + \nu$	$M^{01}, M^{02}, M^{03}, M^{21}, M^{31}, M^{32}$
18	μr^4	$6\mu r^2 + \nu$	$M^{31}, M^{21}, M^{32}, M^{43} - M^{03},$ $M^{42} - M^{02}, M^{41} - M^{01}$

Here μ and ν are arbitrary constants,

$$q_1 = \frac{\alpha x_3}{(r^2 + 1)^2 - 4\tilde{r}^2}, \quad q_2 = \frac{\tilde{r}}{x_3} \left(\frac{r}{r^2 + 1} \right)^3, \quad \varphi = \arctan \frac{x_2}{x_1}, \quad \omega = \alpha \arctan \frac{r^2 + 1}{2x_3} + \varphi.$$

Let us note that solutions presented at Line 18 of Table 2 correspond to Hamiltonian (2) which can be reduced to the Hamiltonian with the constant mass and constant potential. It can be done using the inversion transformation $\mathbf{x} \rightarrow \frac{\mathbf{x}}{r^2}$.

V. EXACT SOLUTIONS FOR MAXIMALLY SUPERINTEGRABLE SYSTEMS

It was conjectured in [46] that all maximally superintegrable systems with two degrees of freedom are exactly solvable, and till now there are no counterexamples for this conjecture. Let us note that for 3d systems the connections between superintegrability and exact solvability is much more complicated.

In this section we show that the maximally superintegrable PDM systems admitting first order integrals of motion are exactly solvable too, and find the corresponding exact solutions explicitly.

A. System invariant w.r.t. algebra $\mathfrak{so}(4)$

Consider Hamiltonian (2) with functions f and V presented in line 16 of Table 2:

$$H = \mu (p_a(1 + r^2)^2 p_a - 6r^2 + \nu). \quad (36)$$

The eigenvalue problem for this Hamiltonian can be written in the following form:

$$\hat{H}\psi \equiv -(\partial_a(1 + r^2)^2 \partial_a + 6r^2)\psi = \tilde{E}\psi \quad (37)$$

where

$$\hat{H} = \frac{1}{\mu}(H - \nu) \quad \text{and} \quad \tilde{E} = \frac{E}{\mu} - \frac{\nu}{\mu}. \quad (38)$$

Equation (37) admits six integrals of motion M^{AB} , $A, B = 1, 2, 3, 4$:

$$\begin{aligned} M^{ab} &= x^a p^b - x^b p^a, \\ M^{4a} &= \frac{1}{2}(r^2 - 1)p^a - x^a x^b p^b + \frac{3i}{2}x^a \end{aligned} \quad (39)$$

which satisfy the following commutation relations:

$$[M^{AB}, M^{CD}] = i(\delta^{AC} M^{BD} + \delta^{BD} M^{AC} - \delta^{AD} M^{BC} - \delta^{BC} M^{AD}) \quad (40)$$

where δ^{AB} is the Kronecker symbol. In other words, operators (39) form a basis of algebra $so(4)$. Defining new basis

$$q_a = \frac{1}{2} \left(M^{4a} + \frac{1}{2} \varepsilon^{abc} M^{bc} \right), \quad g_a = \frac{1}{2} \left(-M^{4a} + \frac{1}{2} \varepsilon^{abc} M^{bc} \right)$$

it is possible to decouple algebra $o(4)$ to the direct sum of two algebras $so(3)$ since q^a and g^a satisfy

$$[q^a, q^b] = i\varepsilon^{abc} q^c, \quad [g^a, g^b] = i\varepsilon^{abc} g^c, \quad [q^a, g^b] = 0.$$

Thus system (37), like the Hydrogen atom, admits six integrals of motion belonging to algebra $so(4)$ and is maximally superintegrable. However, in contrast with the Fock symmetry of the Hydrogen atom, integrals of motion (39) are first order differential operators.

Using the mentioned symmetry it is possible to find eigenvalues \tilde{E} algebraically, before solving equation (37). To do it we calculate Casimir operators of algebra $o(4)$ for representation (39):

$$C_1 = \frac{1}{2} M^{AB} M^{AB} = 2(\mathbf{q}^2 + \mathbf{g}^2) \equiv \frac{1}{4}(\hat{H} - 9), \quad (41)$$

$$C_2 = \frac{1}{2} \varepsilon_{abc} M^{4a} M^{bc} = 2(\mathbf{q}^2 - \mathbf{g}^2) \equiv 0. \quad (42)$$

where $\mathbf{q}^2 = (q^1)^2 + (q^2)^2 + (q^3)^2$, etc.

In accordance with (41) the Hamiltonian eigenvalues can be expressed via eigenvalues of C_1 . Since C_2 is trivial, eigenvalues of C_1 can take the following values (compare with [16], equation (7))

$$C_1 \psi = 4q(q+1)\psi = (n^2 - 1)\psi, \quad n = 1, 2, \dots \quad (43)$$

and so eigenvalues of Hamiltonians H (36) and \tilde{H} are:

$$E = \mu(4n^2 + 5) + \nu, \quad \text{and} \quad \tilde{E} = 4n^2 + 5. \quad (44)$$

The next step is to find eigenvectors of Hamiltonian (36) corresponding to eigenvalues (44). Using the rotation invariance of (37) it is possible to separate variables. Introducing spherical variables and expanding solutions via spherical functions

$$\psi = \frac{1}{r} \sum_{l,m} \phi_{lm}(r) Y_m^l \quad (45)$$

we obtain the following equations for radial functions

$$\left(-(r^2 + 1)^2 \left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) - 4r(r^2 + 1) \frac{\partial}{\partial r} - 2r^2 \right) \varphi_{lm} = (4n^2 + 1) \varphi_{lm}, \quad (46)$$

where $l = 0, 1, 2, \dots$ is the spectral parameter labeling eigenvectors of the squared orbital momentum. The square integrable solutions of these equations are:

$$\varphi_{lm} = C_{lm}^n (r^2 + 1)^{-n-\frac{1}{2}} r^{l+1} \mathcal{F} \left(\left[-n + l + 1, -n + \frac{1}{2} \right], \left[\frac{3}{2} + l \right], -r^2 \right) \quad (47)$$

where $\mathcal{F}(\dots)$ is the hypergeometric function and C_{lm}^n are integration constants. Solutions (47) disappear at infinity provided the first argument of the wave function is a non-positive integer. This condition is in accordance with (44) provided $l \leq n - 1$.

Thus the maximally superintegrable system (37) is exactly solvable. Its exact solutions and spectrum of Hamiltonian (36) are given by equations (47) and (44). Let us note that the sign of eigenvalues E coincides with the sign of coupling constant μ .

B. System invariant w.r.t. algebra $\mathfrak{so}(1,3)$

The next Hamiltonian we consider corresponds to functions f and V presented in line 17 of Table 2:

$$H = \mu (p_a(1 - r^2)^2 p_a - 6r^2 + \nu). \quad (48)$$

The related eigenvalue problem is based on the following equation:

$$\hat{H}\psi \equiv -(\partial_a(1 - r^2)^2 \partial_a + 6r^2) \psi = \tilde{E}\psi \quad (49)$$

where notations (38) are used.

In accordance with Table 2, equation (49) admits six integrals of motion $M^{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$:

$$\begin{aligned} M^{ab} &= x^a p^b - x^b p^a, \quad a, b = 1, 2, 3 \\ M^{0a} &= \frac{1}{2}(r^2 + 1)p^a - x^a x^b p^b + \frac{3i}{2}x^a \end{aligned} \quad (50)$$

which satisfy the following commutation relations:

$$[M^{\mu\nu}, M^{\lambda\sigma}] = i(g^{\mu\lambda} M^{\nu\sigma} + g^{\nu\sigma} M^{\mu\lambda} - g^{\mu\sigma} M^{\nu\lambda} - \delta^{\nu\lambda} M^{\mu\sigma}) \quad (51)$$

where $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

Thus integrals of motion (50) form a basis of algebra $\text{so}(1,3)$. Calculating the corresponding Casimir operators, we obtain:

$$C_1 = \frac{1}{2}M^{ab}M^{ab} - M^{0a}M^{0a} = \frac{1}{4}(\hat{H} + 9), \quad (52)$$

$$C_2 = \frac{1}{2}\varepsilon_{abc}M^{0a}M^{bc} = 0. \quad (53)$$

Like in previous section, the first Casimir operator is proportional to the (shifted) Hamiltonian.

Let us suppose that operators (50) generate a unitary representation (IR) of group $\text{SO}(1,3)$. The admissible eigenvalues c_1 and c_2 of Casimir operators C_1 and C_2 (52) are given by the following formulae [47], [48]:

$$c_1 = 1 - j_0^2 - j_1^2, \quad c_2 = 2ij_0j_1 \quad (54)$$

where j_0 and j_1 are quantum numbers labeling irreducible representations. Since the second Casimir operator C_2 is trivial, we have $c_2 = 0$ and so $j_0 = 0$. In this case there are two possibilities [47]: either j_1 is an arbitrary imaginary number, and the corresponding representation belongs to the principal series, or j_1 is real number satisfying $|j_1| \leq 1$. In the latter case we have a representation belonging to the subsidiary series of IRs. In other words,

$$j_1 = i\lambda, \quad c_1 = 1 - j_1^2 = \lambda^2 + 1 \quad (55)$$

where λ is an arbitrary real number, or, alternatively,

$$0 \leq j_1 \leq 1, \quad c_1 = 1 - j_1^2 \quad (56)$$

In accordance with (52) the related eigenvalues \tilde{E} in (49) can be expressed as:

$$\tilde{E} = -5 - j_1^2. \quad (57)$$

They correspond to the continuous spectrum, and so there are no bound states. Moreover, the possible values of \tilde{E} should belong to the following intervals:

$$-6 \leq \tilde{E} \leq -5 \quad (\text{principal series}) \quad (58)$$

$$-5 \leq \tilde{E} < \infty \quad (\text{subsidiary series}) \quad (59)$$

Thus we can find admissible energies applying purely group-theoretical arguments. Our conclusions can be supported by direct solution of equation (49). Taking into account the

rotational invariance of equation (49) it is convenient to expand its solutions via spherical harmonics, i.e., represent them in form (45). As a result we obtain the following radial equations

$$\left(-(r^2 - 1)^2 \left(\frac{\partial^2}{\partial r^2} - \frac{l(l+1)}{r^2} \right) - 4r(r^2 - 1) \frac{\partial}{\partial r} - 2r^2 \right) \varphi_{lm} = (\tilde{E} + 4) \varphi_{lm}. \quad (60)$$

There is an exceptional point $r = 1$ for equation (60) thus we can expect that its solution will have the corresponding singularity. Indeed, the general solution of (60) looks as follows:

$$\begin{aligned} \varphi_{lm} = & C_{lm}^k (1 - r^2)^{-\frac{1}{2}-k} r^{l+1} \mathcal{F} \left(\left[-k + l + 1, -k + \frac{1}{2} \right], \left[\frac{3}{2} + l \right], r^2 \right) \\ & + \tilde{C}_{lm}^k (1 - r^2)^{-\frac{1}{2}-k} r^{-l} \mathcal{F} \left(\left[-k - l, -k + \frac{1}{2} \right], \left[\frac{1}{2} - l \right], r^2 \right) \end{aligned} \quad (61)$$

where $k = \frac{1}{2} \sqrt{-\tilde{E} - 5}$, and is singular at $r = 1$.

However, setting in (61) $\tilde{C}_{lm}^k = 0$ and $k = j_1$ with j_1 satisfying (58) or (59) we come to solutions which are normalizable in the following metrics:

$$(\psi_1, \psi_2) = \int_0^1 \psi_1^\dagger \psi_2 (r^2 - 1)^3 dr. \quad (62)$$

Moreover, the expression under integral is equal to zero at $r = 0$ and $r = 1$ provided $0 \leq j_1^2 < 1$ or $j_1^2 \leq 0$.

C. Scale invariant system

The last system we consider corresponds to Line 12 of Table 2 where we set $\nu = 0$, and so it is specified by the following Hamiltonian:

$$H = p_a \tilde{r}^2 p_a = -\tilde{r}^2 \Delta - 2x_\alpha \nabla_\alpha, \quad \alpha = 1, 2. \quad (63)$$

This Hamiltonian is transparently invariant with respect to the simultaneous scaling of all independent variables, rotations in the plane 1-2 and shifts of x_3 . Considering the eigenvalue problem for (63) it is convenient to use the cylindrical variables $\tilde{r} = \sqrt{x_1^2 + x_2^2}$, $\varphi = \arctan \frac{x_2}{x_1}$, $x_3 = z$ and expand solutions via eigenfunctions of M^{12} and $P_3 = -i \frac{\partial}{\partial z}$:

$$\Psi = \exp[i(\kappa\varphi + \omega z)] \Phi_{\kappa\omega}(\tilde{r}), \quad \kappa = 0, \pm 1, \pm 2, \dots, -\infty < \omega < \infty.$$

As a result we come to the following equations for radial functions $\Phi = \Phi_{\kappa\omega}(\tilde{r})$:

$$-\tilde{r}^2 \left(\frac{\partial^2 \Phi}{\partial \tilde{r}^2} + \omega^2 \Phi \right) - 3\tilde{r} \frac{\partial \Phi}{\partial \tilde{r}} = (\tilde{E} - \kappa^2) \Phi$$

Square integrable (with the weight \tilde{r}) solutions of this equation are:

$$\Phi_{\kappa\omega} = \frac{1}{\tilde{r}} J_{\alpha}(\omega\tilde{r}), \quad \alpha = \kappa^2 + 1 - \tilde{E} \quad (64)$$

where $J_{\alpha}(\omega\tilde{r})$ is Bessel function of the first kind. Functions (64) are normalizable and disappear at $\tilde{r} = 0$ provided $\alpha \leq 0$. The rescaled energies \tilde{E} continuously take the values $\kappa^2 \leq \tilde{E} \leq \infty$.

VI. DISCUSSION

Thus we find all non-equivalent PDM Schrödinger equations which admit at least one first order integral of motion. These equations are presented by formulae (1) and (2) where f and V are functions collected in Table 2. Equations corresponding to lines 1 – 11 of the table are rather general and include one or two arbitrary functions.

Any of systems whose potentials are presented in lines 1 – 5 admits only one integral of motion. Lines 6 – 8 present integrable equations which admit pairs of commuting integrals of motion. Equations corresponding to lines 9 – 17 admit even more of them and are superintegrable. Moreover, lines 13 – 17 represent maximally superintegrable systems any of which admits four algebraically independent integrals of motion in addition to Hamiltonians (this number is maximal for systems with three degrees of freedom). For completeness in line 18 we present the mass and potential which are equivalent to constant ones.

Let us note that Hamiltonians (36) and (48) are equivalent to particular cases of Hamiltonians considered in [40]–[42], see, for instance, Example 8 in [41] for $A = 0$, or Hamiltonians \hat{H}_{LB} with $\mu = 0$ in [40]. However, just (36) and (48), in contrast with the more general Hamiltonians considered in [40]–[42], admit six first order integrals of motion forming a basis of algebras $\mathfrak{o}(4)$ and $\mathfrak{o}(1,3)$. On the other hand, the superintegrable PDM quantum systems, discussed in [39]–[43] (and other PDM systems) can be obtained in frames of a classification of second order integrals of motion. This work is in progress.

The equivalence group of our classification problem is the $3d$ conformal group whose generators are defined by equation (15). Using this group any of the presented system can be propagated to an entire family of equations. We remind that this group includes the following transformations:

- shifts

$$x_a \rightarrow x'_a = x_a + \nu_a, \quad p_a \rightarrow p_a, \quad \psi \rightarrow \psi' = \psi \quad (65)$$

generated by P_a ;

- rotations

$$x_a \rightarrow R_{ab}x_b, \quad p_a \rightarrow R_{ab}^{-1}p_b, \quad \psi \rightarrow \psi' = \psi \quad (66)$$

where R_{ab} is an orthogonal matrix. This transformation is generated by J_i ;

- dilatation

$$x_a \rightarrow x'_a = e^\lambda x_a, \quad p_a \rightarrow p'_a = e^{-\lambda} p_a, \quad \psi \rightarrow \psi' = e^{-\frac{3\lambda}{2}} \psi \quad (67)$$

generated by D ;

- conformal transformations

$$\begin{aligned} x_a &\rightarrow x'_a = \frac{x_a - \mu_a r^2}{1 - 2\mu_a x_a + \mu^2 r^2}, \\ p_a &\rightarrow p'_a = p_a - 2\mu_a D + 2\varepsilon_{abc}\mu_b J_c + 2\mu_a \mu_b K_b - \mu^2 K_a, \\ \psi &\rightarrow \psi' = \frac{\psi}{(1 - 2\mu_a x_a + \mu^2 r^2)^{\frac{3}{2}}} \end{aligned} \quad (68)$$

where $\mu^2 = \mu_1^2 + \mu_2^2 + \mu_3^2$. These transformations are generated by K_a .

Here ψ is a solution of equation (1), the other Greek letters denote transformation parameters.

Transformations (65)–(68) keep the generic form of equation (1) and algebras of integrals of motion presented in the last column of Table 2. However, they change the explicit forms of functions f and V in Hamiltonian (2) and the corresponding integrals of motion. In other words, any line of Table 2 presents a class of equivalent equations (1) defined up to transformations (65)–(68).

Thus we classify PDM Schrödinger equations which admit the first order integrals of motion. Some of these equations have unusual symmetries being invariant w.r.t. the Lie algebra of Lorentz group $SO(1,3)$ (see section 5.2) and w.r.t. the Lie algebra of group $SO(1,2)$ (see line 10 of Table 2). These "relativistic aspects" of the PDM Schrödinger equations are rather inspiring.

Like the Hydrogen atom, the system discussed in Section 5.1 is invariant w.r.t. algebra $so(4)$. This property can be used to find the Hamiltonian spectra algebraically and to solve equation (37) exactly. Exact solutions of this and some other equations presented in Section 5.

Thus we present the completed list of PDM equations admitting first order integrals of motion. Since such integrals of motion are generators of continuous symmetry groups, our

results can be treated as a group classification of equations (1) including two arbitrary elements, i.e., functions f and V . The natural next step is to extend this result to the case of non-stationary equations including the time derivation. This work is in progress.

One more challenge for researchers is the classification of PDM Hamiltonians admitting higher order integrals of motion. Preliminary results in this field are presented in preprint [49] whose corrected and extended version appeared at March 15, 2015.

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